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Nonlocal Initial Value Problems for Ambartsumian Equation with Hilfer Generalized Proportional Fractional Derivative

S. Manikandan[1], K. Kanagarajan[2], E. M. Elsayed[3] and D. Vivek[4*]

1,2Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641020, India.
3Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
4Department of Mathematics, PSG College of Arts & Science, Coimbatore-641014, India.

E-mail: peppyvivek@gmail.com.

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Abstract

In this research work, we study a new class of fractional type Ambartsumian equation with nonlocal conditions in the sense of Hilfer generalized proportional fractional derivative(HFD). The given problem is first converted into an equivalent fixed point problem, which is then solved by means of the standard fixed point theorems namely, Banach and Krasnosel'skii's fixed point theorem.

Keywords: Ambartsumian equation, Proportional fractional derivative, Existence, Fixed point theorem.

1. Introduction

Fractional calculus is a simplification of ordinary differentiation and integration of arbitrary order, which can be noninteger. Differential equations of fractional order have interested the consideration of several researchers, see [3, 4, 8, 11]. In the works, there exist several definitions of fractional integrals and derivatives, from the most standard Riemann–Liouville and Caputo-type fractional derivatives to the other ones such as Hadamard fractional derivative, the Erdélyi–Kober fractional derivative, and so forth. A generalization of both Riemann–Liouville and Caputo derivatives was given by Hilfer which interpolates between the Riemann–Liouville and Caputo derivatives as it reduces to the Riemann–Liouville and Caputo fractional derivatives. Some properties and applications of the Hilfer derivative can be found in [7] and references therein. The authors in [1, 5, 10] introduced a new type of fractional derivative, the generalized proportional fractional derivative. The work of was generalized in [13, 15] by using the concept of the proportional derivative of a function with respect to another function. In [10], the Hilfer generalized proportional fractional deriva-

* Corresponding author E-mail peppyvivek@gmail.com

tive was proposed. For some recent results on Hilfer generalized proportional fractional differential equations, see [6, 9, 12, 14].

Although several authors developed many interesting techniques and approaches to solve fractional order Ambartsumian equations with their applications. The current articles asserts its novelty from the following perspectives:

(a) We discuss some basic properties and results of generalized HFD.

(b) We obtain the existence of solutions of fractional type Ambartsumian equations via generalized HFD.

2. Axillary Results

We begin this section with important definitions and auxiliary Lemmas that have pertinent to our main results . For more about fractional differential equations, see [1, 2, 3, 4, 5, 7, 8, 13, 15]

Let $-\infty < a < b < \infty$ be finite and infinite intervals on $\mathbb{R}_+$. $C[a,b]$ be the space of the continuous function $\mathbb{Q}$ on $[a,b]$ with the norm defined by

$$\|\mathbb{Q}\|_{C[a,b]} = \max_{t\in[a,b]} |\mathbb{Q}(t)|,$$

and $AC^n[a,b]$, the space of n times absolutely continuous differentiable functions, given by

$$AC^n[a,b] = \left\{\mathbb{Q} : (a,b] \to \mathbb{R}; \mathbb{Q}^{n-1} \in AC([a,b])\right\}.$$

The weighted space $C_\vartheta[a,b]$ of a functions $\mathbb{Q}$ on $[a,b]$ is defined by

$$C_\vartheta[a,b] = \left\{\mathbb{Q} : (a,b] \to \mathbb{R}; (t-a)^\vartheta \mathbb{Q}(t) \in C\left([a,b]\right)\right\}, \quad 0 \le \vartheta \le 1,$$

with the norm

$$\begin{aligned}\|\mathbb{Q}\|_{C_\vartheta[a,b]} &= \left\|(t-a)^\vartheta \mathbb{Q}(t)\right\|_{C_\vartheta[a,b]}, \\ &= \max_{t\in[a,b]} \left|(t-a)^\vartheta \mathbb{Q}(t)\right|.\end{aligned}$$

The weighted space $C^n_\vartheta[a,b]$ of the functions $\mathbb{Q}$ on $(a,b]$ is defined by

$$C^n_\vartheta[a,b] = \left\{\mathbb{Q} : (a,b] \to \mathbb{R}; \mathbb{Q}(t) \in C^{n-1}\left([a,b]\right); \mathbb{Q}^n(t) \in C_\vartheta\left([a,b]\right)\right\}, \quad 0 \le \vartheta < 1,$$

with the norm

$$\|\mathbb{Q}\|_{C^n_\vartheta[a,b]} = \sum_{k=0}^{n-1} \left\|\mathbb{Q}^k\right\|_{C[a,b]} + \|\mathbb{Q}^n\|_{C_\vartheta[a,b]}.$$

Clearly,

$$C^0_\vartheta[a,b] = C_\vartheta[a,b], \quad if \quad n = 0.$$

Definition 2.1. *[11] Suppose $\mathbb{Q} \in \mathcal{L}^1([a,b],\mathbb{R})$. Then the fractional operator*

$$\mathcal{I}_{a^+}^{p}\mathbb{Q}(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\nu)^{(p-1)}\mathbb{Q}(\nu)d\nu, \quad p>0, t>a, n\in\mathbb{N},$$

is referred to as the Riemann-Liouville integral of order p with the lower limit a^+ of the function $\mathbb{Q}$, where $\Gamma(.)$ denotes the classical Gamma function.

Definition 2.2. *[11] Suppose $\mathbb{Q} \in C([a,b],)$. Then the fractional operator*

$${}^{L}\mathcal{D}_{a^+}^{p}\mathbb{Q}(t) = \frac{1}{\Gamma(n-p)}\frac{d^n}{dt^n}\int_a^t (t-\nu)^{(n-p-1)}\mathbb{Q}(\nu)d\nu, \quad p>0, t>a, n-1<p<n, n\in\mathbb{N},$$

is called the Riemann-Liouville fractional derivative of order p with the lower limit a^+ of the function $\mathbb{Q}$, where $\Gamma(.)$ denotes the Gamma function.

Definition 2.3. *[11] Let $\mathbb{Q} \in C^n([a,b])$. Then the fractional operator*

$${}^{C}\mathcal{D}_{a^+}^{p}\mathbb{Q}(t) = \frac{1}{\Gamma(n-p)}\int_a^t (t-\nu)^{(n-p-1)}\mathbb{Q}^n(\nu)d\nu, \quad p>0, t>a, n-1<p<n, n\in\mathbb{N},$$

is referred to as the Caputo fractional derivative of order p with the lower limit a^+ of the function $\mathbb{Q}$.

Definition 2.4. *[10] If $\varrho \in (0,1]$ and $p \in \mathbb{C}, re(p)>0$. Then the fractional operator*

$$\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}(t) = \frac{1}{\varrho^p\Gamma(p)}\int_a^t e^{\frac{\varrho-1}{\varrho}(t-\nu)}(t-\nu)^{(p-1)}\mathbb{Q}(\nu)d\nu, \quad t>a, \tag{1}$$

is called the left-sided generalized proportional integral of order p of the function $\mathbb{Q}$.

Definition 2.5. *[10] The left-sided generalized proportional fractional derivative of order p and $\varrho \in (0,1]$ of a function $\mathbb{Q}$ is defined by*

$$\mathcal{D}_{a^+}^{p,\varrho}\mathbb{Q}(t) = \frac{\mathcal{D}^{n,\varrho}}{\varrho^{n-p}\Gamma(n-p)}\int_a^t e^{\frac{\varrho-1}{\varrho}(t-\nu)}(t-\nu)^{(n-p-1)}\mathbb{Q}(\nu)d\nu, \quad p\in\mathbb{C}, Re(p)>0, \tag{2}$$

where $\Gamma(.)$ is the Gamma function and $n=[p]+1$.

Definition 2.6. *[10] Let $\varrho \in (0,1]$. Then the fractional operator*

$${}^{C}\mathcal{D}_{a^+}^{p,\varrho}\mathbb{Q}(t) = \frac{1}{\varrho^{n-p}\Gamma(n-p)}\int_a^t e^{\frac{\varrho-1}{\varrho}(t-\nu)}(t-\nu)^{(n-p-1)}\left(\mathcal{D}^{n,\varrho}\mathbb{Q}\right)(\nu)d\nu, \quad p\in\mathbb{C}, Re(p)>0, \tag{3}$$

is referred to as the left-sided generalized proportional fractional derivative in the sense of Caputo of order p of the function $\mathbb{Q}$, and $n=[p]+1$.

Certain important properties of the generalized proportional fractional integral and derivative are defined as follows;

Proposition 2.7. *[10] Let $p, \delta \in \mathbb{C}$ such that $Re(p) \geq 0$ and $Re(\delta) > 0$. Then for any $\varrho \in (0,1]$ we have,*

(i) $\left(\mathcal{I}_{a^+}^{p,\varrho} e^{\frac{\varrho-1}{\varrho}s}(s-a)^{\delta-1}\right)(t) = \frac{\Gamma(\delta)}{\varrho^p\Gamma(\delta+p)} e^{\frac{\varrho-1}{\varrho}t}(t-a)^{\delta+p-1}.$

(ii) $\left(\mathcal{D}_{a^+}^{p,\varrho} e^{\frac{\varrho-1}{\varrho}s}(s-a)^{\delta-1}\right)(t) = \frac{\varrho^p\Gamma(\delta)}{\Gamma(\delta-p)} e^{\frac{\varrho-1}{\varrho}t}(t-a)^{\delta-p-1}.$

(iii) $\left(\mathcal{I}_{a^+}^{p,\varrho} e^{\frac{\varrho-1}{\varrho}(b-s)}(b-s)^{\delta-1}\right)(t) = \frac{\Gamma(\delta)}{\varrho^p\Gamma(\delta+p)} e^{\frac{\varrho-1}{\varrho}(b-t)}(b-t)^{\delta+p-1}.$

(iv) $\left(\mathcal{D}_{a^+}^{p,\varrho} e^{\frac{\varrho-1}{\varrho}(b-s)}(b-s)^{\delta-1}\right)(t) = \frac{\varrho^p\Gamma(\delta)}{\Gamma(\delta-p)} e^{\frac{\varrho-1}{\varrho}(b-t)}(b-t)^{\delta-p-1}.$

Theorem 2.8. *[10] Let $\varrho \in (0,1], Re(p) > 0$ and $Re(q) > 0$. If $\mathbb{Q} \in C([a,b],\mathbb{R})$, then*

$$\mathcal{I}_{a^+}^{p,\varrho}\left(\mathcal{I}_{a^+}^{q,\varrho}\mathbb{Q}\right)(t) = \mathcal{I}_{a^+}^{q,\varrho}\left(\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^+}^{p+q,\varrho}\mathbb{Q}\right)(t), \quad t \geq a. \tag{4}$$

Theorem 2.9. *[10] Suppose $\varrho \in (0,1]$ and $0 \leq m < [Re(p)] + 1$. If $\mathbb{Q} \in \mathcal{L}^1([a,b])$. Then*

$$\mathcal{D}_{a^+}^{m,\varrho}\left(\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^+}^{p-m,\varrho}\mathbb{Q}\right)(t), \quad t > a. \tag{5}$$

Corollary 2.10. *[10] If $0 < Re(q) < Re(p)$ and $m - 1 < Re(q) \leq m$. Then we get*

$$\mathcal{D}_{a^+}^{q,\varrho}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}(t) = \mathcal{I}_{a^+}^{p-q,\varrho}\mathbb{Q}(t).$$

Theorem 2.11. *[10] Let $\mathbb{Q} \in \mathcal{L}^1([a,b]), Re(p) > 0$ and $\varrho \in (0,1]$. Then*

$$\mathcal{D}_{a^+}^{p,\varrho}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}(t) = \mathbb{Q}(t), \quad t \geq a, n = [Re(p)] + 1.$$

Lemma 2.12. *[10] If $p > 0, \varrho \in (0,1]$ and $m \in \mathbb{Z}_+$. Then*

$$\left(\mathcal{I}_{a^+}^{p,\varrho}\mathcal{D}_{a^+}^{m,\varrho}\mathbb{Q}\right)(t) = \left(\mathcal{D}_{a^+}^{m,\varrho}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}\right)(t) - \sum_{k=0}^{m-1}\frac{e^{\frac{\varrho-1}{\varrho}(t-a)}(t-a)^{p-m+k}}{\varrho^{p-m+k}\Gamma(p+k-m+1)}\left(\mathcal{D}_{a^+}^{k,\varrho}\mathbb{Q}\right)(a). \tag{6}$$

In particular, if m=1, we obtain

$$\left(\mathcal{I}_{a^+}^{p,\varrho}\mathcal{D}_{a^+}^{\varrho}\mathbb{Q}\right)(t) = \left(\mathcal{D}_{a^+}^{\varrho}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}\right)(t) - \frac{e^{\frac{\varrho-1}{\varrho}(t-a)}(t-a)^{p-1}}{\varrho^{p-1}\Gamma(p)}\mathbb{Q}(a). \tag{7}$$

Theorem 2.13. *[10] Let $Re(p) > 0, n = -[-Re(p)], \mathbb{Q} \in \mathcal{L}^1(a,b)$ and $\left(\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}\right)(t) \in AC^n[a,b]$. Then*

$$\left(\mathcal{I}_{a^+}^{p,\varrho}\mathcal{D}_{a^+}^{p,\varrho}\mathbb{Q}\right)(t) = \mathbb{Q}(t) - e^{\frac{\varrho-1}{\varrho}(t-a)}\sum_{j=1}^{m}\frac{(t-a)^{p-j}}{\varrho^{p-j}\Gamma(p-j+1)}\left(\mathcal{I}_{a}^{j-p,\varrho}\mathbb{Q}\right)(a^+). \tag{8}$$

3. Main Results

In this section, we introduce the HFD and discuss some of its properties.

Definition 3.1. *[11] Let $n-1<p<n, \varrho \in (0,1]$ and $0 \leq q \leq 1$, with $n \in \mathbb{N}$. The left-sided and the right-sided HFD of order p and type q of a function $\mathbb{Q}$ is defined by*

$$\left(\mathcal{D}_{a^{\pm}}^{p,q,\varrho}\mathbb{Q}\right)\mathcal{A} = \mathcal{I}_{a^{\pm}}^{q(n-p),\varrho}\left[\mathcal{D}^{\varrho}\left(\mathcal{I}_{a^{\pm}}^{(1-q)(n-p),\varrho}\mathbb{Q}\right)\right](\mathcal{A}), \tag{9}$$

where $\mathcal{D}^{\varrho}\mathbb{Q}(\mathcal{A}) = (1-\varrho)\mathbb{Q}(\mathcal{A}) + \varrho\mathbb{Q}'(\mathcal{A})$ and $\mathcal{I}$ is the generalized proportional fractional integral defined in definition (2.4).
In particular, if $n=1$, Definition (3.1) is equivalent with

$$\left(\mathcal{D}_{a^{\pm}}^{p,q,\varrho}\mathbb{Q}\right)\mathcal{A} = \mathcal{I}_{a^{\pm}}^{q(1-p),\varrho}\left[\mathcal{D}^{\varrho}\left(\mathcal{I}_{a^{\pm}}^{(1-q)(1-p),\varrho}\mathbb{Q}\right)\right](\mathcal{A}). \tag{10}$$

Thus throughout this paper, we discuss the case where $n=1, 0<p<1, 0 \leq q \leq 1$ and $\vartheta = p+q-pq$.

Remark 3.2.
- *The derivative is used as an interpolator between the Riemann-Liouville and Capudo generalized fractional derivative, respectively, since*

$$\mathcal{D}_{a^{\pm}}^{p,q,\varrho}\mathbb{Q} = \begin{cases} \mathcal{D}^{p}\mathcal{I}_{a^{\pm}}^{(1-p),\varrho}\mathbb{Q}, & q=0 \quad (see \quad Definition 2.5), \\ \mathcal{I}_{a^{\pm}}^{(1-p),\varrho}\mathcal{D}^{p}\mathbb{Q}, & q=1 \quad (see \quad Definition 2.6). \end{cases} \tag{11}$$

- *The parameter ϑ satisfies $0<\vartheta \leq 1, \quad \vartheta \geq p, \quad \vartheta > q, \quad 1-\vartheta < 1-q(1-p)$.*

Lemma 3.3. *The operator $\mathcal{D}_{a^{+}}^{p,q,\varrho}$ can be simplified as*

$$\begin{aligned}\left(\mathcal{D}_{a^{+}}^{p,q,\varrho}\mathbb{Q}\right) &= \mathcal{I}_{a^{+}}^{q(1-p),\varrho}\mathcal{D}^{\varrho}\mathcal{I}_{a^{+}}^{(1-\vartheta),\varrho}\mathbb{Q}, \\ &= \mathcal{I}_{a^{+}}^{q(1-p),\varrho}\mathcal{D}_{a^{+}}^{\vartheta,\varrho}\mathbb{Q},\end{aligned}$$

where $\vartheta = p+q-pq$.

We consider the following weighted spaces of continuous function on $(a,b]$:

$$C_{1-\vartheta}^{p,q}[a,b] = \left\{\mathbb{Q} \in C_{1-\vartheta}[a,b], \mathcal{D}_{a^{+}}^{p,q,\varrho}\mathbb{Q} \in C_{1-\vartheta}[a,b]\right\},$$

and

$$C_{1-\vartheta}^{\vartheta}[a,b] = \left\{\mathbb{Q} \in C_{1-\vartheta}[a,b], \mathcal{D}_{a^{+}}^{\vartheta,\varrho}\mathbb{Q} \in C_{1-\vartheta}[a,b]\right\}.$$

Since $\mathcal{D}_{a^{+}}^{p,q,\varrho} = \mathcal{I}_{a^{+}}^{q(1-p),\varrho}\mathcal{D}_{a^{+}}^{\vartheta,\varrho}$,
$C_{1-\vartheta}^{\vartheta}[a,b] \subset C_{1-\vartheta}^{p,q}[a,b]$.

Lemma 3.4. *Suppose* $0 < p < 1, \varrho \in (0,1]$ *and* $0 \leq \vartheta < 1$. *If* $\mathbb{Q} \in C_\vartheta[a,b]$, *then*

$$\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}(a) = \lim_{t \to a^+} \mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}(t) = 0, \quad 0 \leq \vartheta < p.$$

Lemma 3.5. *Let* $0 < p < 1, \varrho \in (0,1], 0 \leq q \leq 1$ *and* $\vartheta = p + q - pq$. *If* $\mathbb{Q} \in C_{1-\vartheta}^{\vartheta}[a,b]$ *then*

$$\mathcal{I}_{a^+}^{\vartheta,\varrho}\mathcal{D}_{a^+}^{\vartheta,\varrho}\mathbb{Q} = \mathcal{I}_{a^+}^{p,\varrho}\mathcal{D}_{a^+}^{p,q,\varrho}\mathbb{Q}$$

and

$$\mathcal{D}_{a^+}^{\vartheta,\varrho}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q} = \mathcal{D}_{a^+}^{q(1-p),\varrho}\mathbb{Q}.$$

Lemma 3.6. *Suppose* $\mathbb{Q} \in \mathcal{L}^1(a,b)$ *such that* $\mathcal{D}_{a^+}^{q(1-p),\varrho}\mathbb{Q}$ *exists in* $\mathcal{L}^1(a,b)$. *Then*

$$\mathcal{D}_{a^+}^{p,q,\varrho}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q} = \mathcal{I}_{a^+}^{q(1-p),\varrho}\mathcal{D}_{a^+}^{q(1-p),\varrho}\mathbb{Q}.$$

Lemma 3.7. *Let* $0 < p < 1, \varrho \in (0,1], 0 \leq \vartheta < 1$. *If* $\mathbb{Q} \in C_\vartheta[a,b]$ *and then* $\mathcal{I}_{a^+}^{(1-p),\varrho}\mathbb{Q} \in C_\vartheta^1[a,b]$, *then*

$$\mathcal{I}_{a^+}^{p,\varrho}\mathcal{D}_{a^+}^{p,\varrho}\mathbb{Q}(\mathcal{A}) = \mathbb{Q}(\mathcal{A}) - e^{\frac{\varrho-1}{\varrho}(\mathcal{A}-a)}\frac{(\mathcal{A}-a)^{p-1}}{\varrho^{p-1}\Gamma(p)}\left(\mathcal{I}_a^{(1-p),\varrho}\right)(a^+), \quad \forall \mathcal{A} \in (a,b].$$

Lemma 3.8. *Let* $0 < p < 1, \varrho \in (0,1], 0 \leq q \leq 1$ *and* $\vartheta = p + q - pq$. *If* $\mathbb{Q} \in C_{1-\vartheta}[a,b]$ *and* $\mathcal{D}_{a^+}^{p,q,\varrho}\mathbb{Q}$ *then* $\mathcal{D}_{a^+}^{p,q,\varrho}\mathbb{Q}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}$ *exists in* (a,b) *and*

$$\mathcal{D}_{a^+}^{p,q,\varrho}\mathcal{I}_{a^+}^{p,\varrho}\mathbb{Q}(\mathcal{A}) = \mathbb{Q}(\mathcal{A}), \quad \mathcal{A} \in (a,b].$$

Lemma 3.9. *Let* $0 < p < 1, \varrho \in (0,1], 0 \leq q \leq 1$ *and* $0 < \vartheta < 1$. *If* $\mathbb{Q} \in C_{1-\vartheta}[a,b]$ *and* $\mathcal{I}_{a^+}^{1-\vartheta,\varrho}\mathbb{Q}$, *then*

$$\mathcal{I}_{a^+}^{p,\varrho}\mathcal{D}_{a^+}^{p,q,\varrho}\mathbb{Q}(\mathcal{A}) = \mathbb{Q}(\mathcal{A}) - e^{\frac{\varrho-1}{\varrho}(\mathcal{A}-a)}\frac{(\mathcal{A}-a)^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)}\left(\mathcal{I}_a^{(1-\vartheta),\varrho}\mathbb{Q}\right)(a^+), \quad \forall \mathcal{A} \in (a,b].$$

4. Existence Theory

Ambartsumian derived the standard Ambartsumian equation. The absorption of light by interstellar matter has been defined in this equation. In the theory of surface brightness in the Milky Way, the Ambartsumian delay equation is used, see [2].

Let us consider the following fractional differential equation

$$\begin{cases} \mathcal{D}_{a^+}^{p,q,\varrho}\mathcal{A}(t) = \mathbb{Q}(t, \mathcal{A}(t)) = \mathbb{Q}\left(t, \mathcal{A}(\frac{t}{\eta}), \mathcal{A}(t)\right), \quad t \in J = [a,T], T > a \geq 0, \\ \mathcal{I}_{a^+}^{1-\vartheta,\varrho}\mathcal{A}(a) = \sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i), \quad p \leq \vartheta = p + q - pq, \tau_i \in (a,T), \end{cases} \tag{12}$$

where $\mathcal{D}_{a^+}^{p,q,\varrho}(.)$ is the HFD of order p such that $0 < p < 1$, $\mathcal{I}_{a^+}^{1-\vartheta,\varrho}(.)$ is the generalized proportional fractional integral of order $1-\vartheta$ such that $1-\vartheta > 0, \mu_i \in \mathbb{R}, \mathbb{Q} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\tau_i \in J$ satisfying $a < \tau_1 < .....\tau_m < T$ for $i = 1, 2, ..., m$.

Theorem 4.1. *Let $0 < p < 1, 0 \leq q \leq 1$ and $\vartheta = p + q - pq$ and let $\mathbb{Q} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbb{Q} \in C_{1-\vartheta}[J, \mathbb{R}]$ for any $\mathcal{A} \in C_{1-\vartheta}[J, \mathbb{R}]$. If $\mathcal{A} \in C^{\vartheta}_{1-\vartheta}[J, \mathbb{R}]$ then $\mathcal{A}$ satisfies the problem (12) if and only if $\mathcal{A}$ satisfies the mixed type integral equation*

$$\mathcal{A}(t) = \frac{\wedge}{\varrho^p \Gamma(p)} e^{\frac{\varrho-1}{\varrho}(t-a)} (t-a)^{\vartheta-1} \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\tau_i - s)} (\tau_i - s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds$$

$$+ \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds, \tag{13}$$

where,

$$\wedge = \frac{1}{\varrho^{\vartheta-1}\Gamma(\vartheta) - \sum_{i=1}^{m} \mu_i e^{\frac{\varrho-1}{\varrho}(\tau_i - a)} (\tau_i - a)^{\vartheta-1}}. \tag{14}$$

Proof. Suppose $\mathcal{A} \in C^{\vartheta}_{1-\vartheta}[J, \mathbb{R}]$ be a solution of (12). We have to prove that $\mathcal{A}$ is a solution of our proposed problem (13). By the Lemma 3.9, we have

$$\mathcal{A}(t) = \tfrac{(t-a)^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)} e^{\frac{\varrho-1}{\varrho}(t-a)} \mathcal{I}_{a^+}^{1-\vartheta,\varrho} \mathcal{A}(a^+)$$
$$+ \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds. \tag{15}$$

Now substituting $t = \tau_i$ and multiplying μ_i on the both sides of the equation (15),we get

$$\mu_i \mathcal{A}(\tau_i) = \frac{(\tau_i - a)^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)} e^{\frac{\varrho-1}{\varrho}(\tau_i - a)} \mu_i \mathcal{I}_{a^+}^{1-\vartheta,\varrho} \mathcal{A}(a^+) + \mu_i \mathcal{I}_{a^+}^{p,\varrho} \mathbb{Q}(\tau_i), \tag{16}$$

which implies that

$$\sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i) = \frac{1}{\varrho^{\vartheta-1}\Gamma(\vartheta)} \sum_{i=1}^{m} \mu_i e^{\frac{\varrho-1}{\varrho}(\tau_i - a)} (\tau_i - a)^{\vartheta-1} \mathcal{I}_{a^+}^{1-\vartheta,\varrho} \mathcal{A}(a^+)$$

$$+ \frac{1}{\varrho^p \Gamma(p)} \sum_{i=1}^{m} \mu_i \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds. \tag{17}$$

From the initial condition $\mathcal{I}_{a^+}^{1-\vartheta,\varrho} \mathcal{A}(a) = \sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i)$, we get

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho} \mathcal{A}(a^+) = \frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p \Gamma(p)} \wedge \sum_{i=1}^{m} \mu_i \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds. \tag{18}$$

Therefore the result follows by substituting Eq. (18) in Eq. (15). This shows that $\mathcal{A}(t)$ satisfies the Eq.(13).

Conversely, suppose that $\mathcal{A} \in C^{\vartheta}_{1-\vartheta}$ Satisfies the Eq.(13), then we have to show that $\mathcal{A}$ also satisfies the Eq.(12). Applying $\mathcal{D}_{a^+}^{\vartheta,\varrho}$ on the both sides of the Eq.(13).

$\mathcal{D}_{a^+}^{\vartheta,\varrho} \mathcal{A}(t)$

$$= \mathcal{D}_{a^+}^{\vartheta,\varrho} \left(\tfrac{\wedge}{\varrho^p \Gamma(p)} e^{\frac{\varrho-1}{\varrho}(t-a)} (t-a)^{\vartheta-1} \textstyle\sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\tau_i - s)} (\tau_i - s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\tfrac{s}{\eta}\right), \mathcal{A}(s)\right) ds \right)$$

$$+\mathcal{D}_{a^+}^{\vartheta,\varrho}\left(\frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)ds\right).$$

By the Proposition(2.7), Theorem(2.9) and Definition(3.1), we get

$$\mathcal{D}_{a^+}^{\vartheta,\varrho}\mathcal{A}(t)=\left(\mathcal{D}_{a^+}^{q(1-p),\varrho}\mathbb{Q}\left(t,\mathcal{A}(\frac{t}{\eta}),\mathcal{A}(t)\right)\right)(\mathcal{A}). \tag{19}$$

Since $\mathcal{D}_{a^+}^{p,q,\varrho}\mathcal{A}\in C_{1-\vartheta}[J,\mathbb{R}]$, by the definition of $C_{1-\vartheta}^{\vartheta}[J,\mathbb{R}]$, Eq.(19) implies that

$$\mathcal{D}_{a^+}^{q(1-p),\varrho}\mathbb{Q}=\mathcal{D}^{\varrho}\mathcal{I}_{a^+}^{1-q(1-p),\varrho}\mathbb{Q}\in C_{1-\vartheta,\varrho}[J,\mathbb{R}].$$

For $\mathbb{Q}\in C_{1-\vartheta}[J,\mathbb{R}]$ and from Theorem(2.11), we can see that $\mathcal{I}_{a^+}^{1-q(1-p),\varrho}\mathbb{Q}\in C_{1-\vartheta,\varrho}[J,\mathbb{R}]$, this implies that $\mathcal{I}_{a^+}^{1-q(1-p),\varrho}\mathbb{Q}\in C_{1-\vartheta}^{1}[J,\mathbb{R}]$ from the definition of $C_{1-\vartheta}^{n}[J,\mathbb{R}]$.
Applying $\mathcal{I}_{a^+}^{q(1-p),\varrho}$ on both sides of the equation(19) and by the Proposition(2.7), Lemma(3.7) and definition(3.1)

$$\begin{aligned}\mathcal{I}_{a^+}^{q(1-p),\varrho}\mathcal{D}_{a^+}^{\vartheta,\varrho}\mathcal{A}(t)&=\mathcal{I}_{a^+}^{q(1-p),\varrho}\left(\mathcal{D}_{a^+}^{q(1-p),\varrho}\mathbb{Q}\left(t,\mathcal{A}(\frac{t}{\eta}),\mathcal{A}(t)\right)\right),\\&=\mathbb{Q}\left(t,\mathcal{A}(\frac{t}{\eta}),\mathcal{A}(t)\right)-\frac{\mathcal{I}_{a^+}^{q(1-p),\varrho}\mathbb{Q}(a)}{\Gamma q(1-p)}(t-a)^{q(p-1)-1},\end{aligned}$$

$$\mathcal{I}_{a^+}^{q(1-p),\varrho}\mathcal{D}_{a^+}^{\vartheta,\varrho}\mathcal{A}(t)=\mathbb{Q}\left(t,\mathcal{A}(\frac{t}{\eta}),\mathcal{A}(t)\right). \tag{20}$$

Hence its remains to show that if $\mathcal{A}\in C_{1-\vartheta}^{\vartheta}[J,\mathbb{R}]$ satisfies the Eq.(13), it also satisfies the initial condition. So applying $\mathcal{I}_{a^+}^{1-\vartheta,\varrho}$ on the both sides of the Eq.(13) and by the Proposition(2.7), Theorem(2.8) and Corollary(2.10), we obtain

$$\begin{aligned}&\mathcal{I}_{a^+}^{1-\vartheta,\varrho}\mathcal{A}(t)\\&=\mathcal{I}_{a^+}^{1-\vartheta,\varrho}\left(\frac{\wedge}{\varrho^p\Gamma(p)}e^{\frac{\varrho-1}{\varrho}(t-a)}(t-a)^{\vartheta-1}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\tau_i-s)}(\tau_i-s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)ds\right)\\&+\mathcal{I}_{a^+}^{1-\vartheta,\varrho}\left(\frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)ds\right)\end{aligned}$$

$$\begin{aligned}&\mathcal{I}_{a^+}^{1-\vartheta,\varrho}\mathcal{A}(t)\\&=\frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p\Gamma(p)}\wedge e^{\frac{\varrho-1}{\varrho}(t-a)}(t-a)^{\vartheta-1}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\tau_i-s)}(\tau_i-s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)ds\\&\qquad+\mathcal{I}_{a^+}^{1-q(1-p),\varrho}\mathbb{Q}(t).\end{aligned} \tag{21}$$

Taking the limit as $t\to a^+$ in Eq.(21) and the fact that $1-q<1-p(1-r)$ gives

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho}\mathcal{A}(a^+)=\frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p\Gamma(p)}\wedge\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\tau_i-s)}(\tau_i-s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)ds. \tag{22}$$

Substituting $t = \tau_i$ and multiplying μ_i in Eq.(13), we get

$$\mu_i \mathcal{A}(\tau_i)$$
$$= \frac{\wedge}{\varrho^p \Gamma(p)} e^{\frac{\varrho-1}{\varrho}(\tau_i - a)} (\tau_i - a)^{\vartheta-1} \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\tau_i - s)} (\tau_i - s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds$$
$$+ \frac{\mu_i}{\varrho^p \Gamma(p)} \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\tau_i - s)} (\tau_i - s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds,$$

which implies that

$$\sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i) = \wedge \sum_{i=1}^{m} \mu_i \mathcal{I}_{a^+}^{p,\varrho} \mathbb{Q}(\tau_i) \sum_{i=1}^{m} \mu_i e^{\frac{\varrho-1}{\varrho}(\tau_i - a)} (\tau_i - a)^{\vartheta-1} + \sum_{i=1}^{m} \mu_i \mathcal{I}_{a^+}^{p,\varrho} \mathbb{Q}(\tau_i),$$

$$\sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i) = \sum_{i=1}^{m} \mu_i \mathcal{I}_{a^+}^{p,\varrho} \mathbb{Q}(\tau_i) \left(1 + \sum_{i=1}^{m} \mu_i e^{\frac{\varrho-1}{\varrho}(\tau_i - a)} (\tau_i - a)^{\vartheta-1}\right). \tag{23}$$

Thus,

$$\sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i) = \frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p \Gamma(p)} \wedge \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\tau_i - s)} (\tau_i - s)^{p-1} \mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds. \tag{24}$$

So by Eq.(22) and Eq.(24), we get

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho} \mathcal{A}(t) = \sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i). \tag{25}$$

Hence the proof is completed. □

Next we have to prove the uniqueness of solutions of the problem(12) using the concepts of the Banach contraction principle. To demonstrate our main result, the following must be satisfied.

(H_1): Let $\mathbb{Q} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbb{Q} \in C_{1-\vartheta}^{q(1-p)}[J, \mathbb{R}]$ for any $\mathcal{A} \in C_{1-\vartheta}^{\vartheta}[J, \mathbb{R}]$.
(H_2): There exists a constant $K > 0$ such that

$$|\mathbb{Q}(t, u) - \mathbb{Q}(t, \overline{u})| \leq K |u, \overline{u}|, \quad forany \quad u, \overline{u} \in \mathbb{R} \quad and \quad t \in J.$$

(H_3): Suppose that $K\psi < 1$,
where

$$\psi = \frac{\beta(\vartheta, p)}{\varrho^p \Gamma(p)} \left(|\wedge| \sum_{i=1}^{m} \mu_i (\tau_i - a)^{p+\vartheta-1} + (T - a)^p\right) \tag{26}$$

and $\beta(\vartheta, p)$ is the beta function defined by (See.[11])

$$\beta(\vartheta, p) = \int_0^1 \mathcal{A}^{\vartheta,p} (1 - \mathcal{A})^{p-1} d\mathcal{A}, \quad Re(\vartheta), Re(p) > 0.$$

Theorem 4.2. *Let $0 < p < 1, 0 \leq q \leq 1$ and $\vartheta = p + q - pq$. Suppose that the hypotheses $(H_1) - (H_3)$ are satisfied. Then the problem(12) has a unique solution in the space $C^{\vartheta}_{1-\vartheta}[J, \mathbb{R}]$.*

Proof. Define the operator $\mathcal{T} : C_{1-\vartheta}[J, \mathbb{R}] \to C_{1-\vartheta}[J, \mathbb{R}]$ by

$$(\mathcal{T}\mathcal{A})(t) = \frac{\wedge}{\varrho^p\Gamma(p)} e^{\frac{\varrho-1}{\varrho}(t-a)}(t-a)^{\vartheta-1} \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\tau_i-s)}(\tau_i-s)^{p-1}\mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds$$
$$+\frac{1}{\varrho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{p-1}\mathbb{Q}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) ds. \tag{27}$$

It follows the operator $\mathcal{T}$ is well defined. Now for any $\mathcal{A}_1, \mathcal{A}_2 \in C_{1-\vartheta}[J, \mathbb{R}]$ and $t \in J$, this gives

$$\begin{aligned}
&\left|\left((\mathcal{T}\mathcal{A}_1)(t) - (\mathcal{T}\mathcal{A}_2)(t)\right)(t-a)^{1-\vartheta}\right| \\
&\leq \frac{|\wedge|}{\varrho^p\Gamma(p)} \left|e^{\frac{\varrho-1}{\varrho}(t-a)}(t-a)^{\vartheta-1}\right| \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} \left|e^{\frac{\varrho-1}{\varrho}(\tau_i-s)}\right| (\tau_i-s)^{p-1} \\
&\left|\mathbb{Q}\left(s, \mathcal{A}_1\left(\frac{s}{\eta}\right), \mathcal{A}_1(s)\right) - \mathbb{Q}\left(s, \mathcal{A}_2\left(\frac{s}{\eta}\right), \mathcal{A}_2(s)\right)\right| ds + \frac{1}{\varrho^p\Gamma(p)} \\
&\int_a^t \left|e^{\frac{\varrho-1}{\varrho}(t-s)}\right| (t-s)^{p-1} \left|\mathbb{Q}\left(s, \mathcal{A}_1\left(\frac{s}{\eta}\right), \mathcal{A}_1(s)\right) - \mathbb{Q}\left(s, \mathcal{A}_2\left(\frac{s}{\eta}\right), \mathcal{A}_2(s)\right)\right| ds.
\end{aligned} \tag{28}$$

Since $\left|e^{\frac{\varrho-1}{\varrho}t}\right| < 1$, we get

$$\begin{aligned}
&\left|\left((\mathcal{T}\mathcal{A}_1)(t) - (\mathcal{T}\mathcal{A}_2)(t)\right)(t-a)^{1-\vartheta}\right| \\
&\leq \frac{K|\wedge|}{\varrho^p\Gamma(p)} \left(\sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} (\tau_i-s)^{p-1}(s-a)^{\vartheta-1} ds\right) \|\mathcal{A}_1 - \mathcal{A}_2\|_{C_{1-\vartheta}[J,\mathbb{R}]} \\
&+ \frac{K}{\varrho^p\Gamma(p)}(t-a)^{1-\vartheta} \left(\int_{a^+}^{t} (t-s)^{p-1}(s-a)^{\vartheta-1} ds\right) \|\mathcal{A}_1 - \mathcal{A}_2\|_{C_{1-\vartheta}[J,\mathbb{R}]} \\
&\leq \frac{K|\wedge|}{\varrho^p\Gamma(p)}\beta(\vartheta, p) \sum_{i=1}^{m} \mu_i(\tau_i-s)^{p+\vartheta-1} \|\mathcal{A}_1 - \mathcal{A}_2\|_{C_{1-\vartheta}[J,\mathbb{R}]} \\
&+ \frac{K}{\varrho^p\Gamma(p)}(T-a)^{p}\beta(\vartheta, p) \|\mathcal{A}_1 - \mathcal{A}_2\|_{C_{1-\vartheta}[J,\mathbb{R}]}.
\end{aligned}$$

Therefore,

$$\|(\mathcal{T}\mathcal{A}_1) - (\mathcal{T}\mathcal{A}_2)\| |_{C_{1-\vartheta}[J,\mathbb{R}]} \leq K\psi \|\mathcal{A}_1 - \mathcal{A}_2\|_{C_{1-\vartheta}[J,\mathbb{R}]}. \tag{29}$$

Hence it follows from the Eq.(26) that $\mathcal{T}$ is a contraction map. Thus, as a consequence of the Banach contraction principle, problem (12) has a unique solution. □

Now we have to prove that the existence of the solution of our proposed problem (12) using the concepts of Krasnoselskii's fixed point theorem (See.[?]). We consider the following hypotheses

(H_4) : Suppose that $K\triangle < 1$,

where,

$$\triangle = \frac{\beta(\vartheta,p)}{\varrho^p\Gamma(p)}\left|\wedge\right|\sum_{i=1}^{m}\mu_i(\tau_i-a)^{p+\vartheta-1}. \tag{30}$$

Theorem 4.3. *Let $0<p<1, 0\le q\le 1$ and $\vartheta=p+q-pq$. Suppose that the hypotheses $(H_1),(H_2)$ and (H_4) are satisfied. Then the problem (12) has at least one solution in the space $C^{\vartheta}_{1-\vartheta}[J,\mathbb{R}]$.*

Proof. We have $\|\alpha\|_{C_{1-\vartheta}[J,\mathbb{R}]} = \sup_{t\in J}\left|(t-a)^{1-\vartheta}\alpha(t)\right|$ and choose $k \ge M\|\alpha\|_{C_{1-\vartheta}[J,\mathbb{R}]}$, where

$$M = \frac{\beta(\vartheta,p)}{\varrho^p\Gamma(p)}\left(\left|\wedge\right|\sum_{i=1}^{m}\mu_i(\tau_i-a)^{p+\vartheta-1}+(T-a)^p\right). \tag{31}$$

We consider $B_k = \left\{x\in\mathbb{C}[J,\mathbb{R}] : \|\mathcal{A}\|_{C_{1-\vartheta}[J,\mathbb{R}]} < k\right\}$.

Define the operators $\mathcal{G}$ and $\mathcal{H}$ on B_k by

$$\mathcal{G}\mathcal{A}(t) = \frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)ds,$$

$$\mathcal{H}\mathcal{A}(t) = \frac{\wedge}{\varrho^p\Gamma(p)}e^{\frac{\varrho-1}{\varrho}(t-a)}(t-a)^{\vartheta-1}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\tau_i-s)}(\tau_i-s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)ds,$$

for all $t\in[a,T]$.

Now, for every $\mathcal{A},\overline{\mathcal{A}}\in B_k$,

$$\begin{aligned}
&\left|\left(\mathcal{G}\mathcal{A}(t)+\mathcal{H}\overline{\mathcal{A}}(t)\right)(t-a)^{1-\vartheta}\right| \\
&\le \frac{(t-a)^{1-\vartheta}}{\varrho^p\Gamma(p)}\int_a^t(t-s)^{p-1}(s-a)^{\vartheta-1}\left|\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)(s-a)^{1-\vartheta}\right|ds \\
&+\frac{|\wedge|}{\varrho^p\Gamma(p)}\sum_{i=1}^{m}\mu_i\int_a^{\tau_i}(\tau_i-s)^{p-1}(\tau_i-a)^{\vartheta-1}\left|\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right)(\tau_i-a)^{1-\vartheta}\right|ds, \\
&\le \|\alpha\|\left[\frac{\beta(\vartheta,p)}{\varrho^p\Gamma(p)}\left|\wedge\right|\sum_{i=1}^{m}\mu_i(\tau_i-a)^{p+\vartheta-1}+\frac{\beta(\vartheta,p)}{\varrho^p\Gamma(p)}(T-a)^p\right], \\
&\le \|\alpha\| M,
\end{aligned}$$

$$\left|\left(\mathcal{G}\mathcal{A}(t)+\mathcal{H}\overline{\mathcal{A}}(t)\right)(t-a)^{1-\vartheta}\right| \le k < \infty. \tag{32}$$

This implies that $\mathcal{G}\mathcal{A}(t)+\mathcal{H}\overline{\mathcal{A}}(t)\in B_k$.

Now we have to prove that $\mathcal{H}$ is a contraction.

Now, let $\mathcal{A}, \overline{\mathcal{A}} \in C_{1-\vartheta}[J,\mathbb{R}]$ and $t \in J$, then

$$\left|\left(\mathcal{H}\mathcal{A}(t) + \mathcal{H}\overline{\mathcal{A}}(t)\right)(t-a)^{1-\vartheta}\right|$$

$$\begin{aligned}
&= \left|\wedge e^{\frac{\varrho-1}{\varrho}}(t-a)\sum_{i=1}^{m}\mu_i \mathcal{I}_{a^+}^{p,\varrho}\left(\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right) - \mathbb{Q}\left(s,\overline{\mathcal{A}}\left(\frac{s}{\eta}\right),\overline{\mathcal{A}}(s)\right)\right)(\tau_i)\right|,\\
&\le \frac{K\left|\wedge\right|}{\varrho^p\Gamma(p)}\sum_{i=1}^{m}\mu_i\int_a^{\tau_i}(\tau_i - s)^{p-1}(\tau_i - s)^{\vartheta-1}\left|\mathcal{A}(s) - \overline{\mathcal{A}}(s)\right| ds,\\
&\le \left[\frac{K\left|\wedge\right|}{\varrho^p\Gamma(p)}\beta(\vartheta,p)\sum_{i-1}^{m}\mu_i(\tau_i - a)^{p+\vartheta-1}\right]\left|\mathcal{A}(s) - \overline{\mathcal{A}}(s)\right|_{C_{1-\vartheta}[J,\mathbb{R}]},\\
&\le K\triangle\left|\mathcal{A}(s) - \overline{\mathcal{A}}(s)\right|_{C_{1-\vartheta}[J,\mathbb{R}]}.
\end{aligned}$$

Hence it follows from (H_4) that $\mathcal{H}$ is a contraction.

We have to show that the operator $\mathcal{G}$ is continuous and compact.

Clearly, the operator $\mathcal{G}$ is continuous, due to the fact that the function $\mathbb{Q}$ is continuous.

Thus, for any $\mathcal{A} \in C_{1-\vartheta}[J,\mathbb{R}]$, we have

$$\|\mathcal{G}\mathcal{A}\| \le \|\eta\| \frac{\beta(\vartheta,p)}{\varrho^p\Gamma(p)}(T-a)^p < \infty.$$

This shows that the operator $\mathcal{G}$ is uniformly bounded on B_k. Thus, it remains to shows that $\mathcal{G}$ is compact. Denoting $\sup_{(t,x)\in J\times B_k}\left|\mathbb{Q}\left(t,\mathcal{A}\left(\frac{t}{\eta}\right),\mathcal{A}(t)\right)\right| = \delta < \infty$ and let for $a < \tau_1 < \tau_2 < T$,

$$\left|(\tau_2 - a)^{1-\vartheta}\left(\mathcal{G}\mathcal{A}(\tau_2)\right) + (\tau_1 - a)^{1-\vartheta}\left(\mathcal{G}\mathcal{A}(\tau_1)\right)\right|$$

$$\begin{aligned}
&= \left|\frac{(\tau_2 - a)^{1-\vartheta}}{\varrho^p\Gamma(p)}\int_a^{\tau_2} e^{\frac{\varrho-1}{\varrho}(\tau_2 - s)}(\tau_2 - s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right) ds\right.\\
&\quad \left. - \frac{(\tau_1 - a)^{1-\vartheta}}{\varrho^p\Gamma(p)}\int_a^{\tau_1} e^{\frac{\varrho-1}{\varrho}(\tau_1 - s)}(\tau_1 - s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right) ds\right|.
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{\varrho^p\Gamma(p)}\int_a^{\tau_2}[(\tau_2 - a)^{1-\vartheta}(\tau_2 - s)^{p-1} - (\tau_1 - a)^{1-\vartheta}(\tau_1 - s)^{p-1}]\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right) ds\\
&\quad + \frac{1}{\varrho^p\Gamma(p)}\int_{\tau_1}^{\tau_2}(\tau_2 - a)^{1-\vartheta}(\tau_2 - s)^{p-1}\mathbb{Q}\left(s,\mathcal{A}\left(\frac{s}{\eta}\right),\mathcal{A}(s)\right) ds.
\end{aligned}$$

$$\left|(\tau_2 - a)^{1-\vartheta}\left(\mathcal{G}\mathcal{A}(\tau_2)\right) + (\tau_1 - a)^{1-\vartheta}\left(\mathcal{G}\mathcal{A}(\tau_1)\right)\right| \to 0 \quad as \quad \tau_1 \to \tau_2.$$

As a consequence of Arzelá-Ascoli theorem, the operator $\mathcal{G}$ is Compact on B_k. Thus, problem (12) has at least one solution. □

5. Conclusion

We have successfully studied fractional type Ambartsumian equations with nonlocal initial conditions, using the generalized HFD. Also, we have provided some sufficient conditions guaranteeing the existence of solutions for a class of fractional order Ambartsumian equations. We will apply the numerical algorithms to the proposed problems with further scope.

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The authors declare no conflicts of interest.

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